A simple proof of Bourgain's theorem on the singularity of the spectrum of Ornstein's maps *

E. H. el Abdalaoui†

October 2, 2014

Abstract. We give a simple proof of Bourgain's theorem on the singularity of Ornstein's maps.

Setting and proof. Let (m_j), (t_j) be a sequence of positive integers, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space associated to Ornstein's construction, that is,

$$\Omega = \prod_{j=1}^{+\infty} \big\{ -t_j, \cdots, t_j \big\}^{p_j - 1}, \quad \mathbb{P} = \bigotimes_{j=1}^{+\infty} \bigotimes_{k=1}^{p_j-1} \mathcal{U}_k,$$

where $\mathcal{U}_k$ is the uniform measure on $\big\{ -t_j, \cdots, t_j \big\}^{p_j-1}$.

We want to prove the following theorem due to Bourgain [2].

For almost all $\omega \in \Omega$, the spectral type μ_ω of the rank one map T_ω is singular.

Recall that μ_ω is the weak-star limit of the following sequence of probability measures

$$\Big(\prod_{j=1}^{N} |P_j(\omega, z)|^2 d\lambda \Big)_{N \geq 1},$$

where λ is the Lebesgue measure and for each $j \in \mathbb{N}^*$,

$$P_j(z) = \frac{1}{\sqrt{m_j}} \sum_{k=0}^{m_j - 1} z^{n_{j,k}(\omega)},$$

$$n_{j,0}(\omega) = 0 \quad \text{and} \quad \text{for } \; k \geq 1, n_{j,k}(\omega) = k(h_j + t_j) + x_{j,k}(\omega) = k(h_j + t_j) + \omega_{j,k}.$$

We further assume that the sequence (m_j) is unbounded. Therefore, by Theorem 5.2 in [1]. combined with the uniform integrability of the sequence $\prod_{j=1}^{N} |P_j(\omega, z)|$, we have

$$\int_\Omega \int \prod_{j=1}^{N} |P_j(\omega, z)| dz d\mathbb{P} \xrightarrow[N \to +\infty]{} \int_\Omega \int \sqrt{\frac{d\mu_\omega}{d\lambda}} dz d\mathbb{P},$$

We further have

$$\lim_{j \longrightarrow +\infty} \int |P_j(\omega, z)| d\mathbb{P} dz = \lim_{j \longrightarrow +\infty} \int |\widetilde{P}_j(\omega, z)| d\mathbb{P} dz = \frac{1}{2}\sqrt{\pi}, \quad \text{with} \quad \widetilde{P}_j(\omega, z) = P_j(\omega, z) - \int P_j(\omega, z) d\mathbb{P}.$$

The last equality follows from the classical Lindeberg's central limit theorem (CLT) [1] combined with Lebesgue dominated convergence theorem and the uniform integrability of the sequence $(|P_j(\omega, z)|)_{j \geq 0}$ under $dz \otimes \mathbb{P}$. We thus get

$$\int \int_\Omega \prod_{j=1}^{N} |P_j(\omega, z)| d\mathbb{P} dz = \int \prod_{j=1}^{N} \int_\Omega |P_j(\omega, z)| d\mathbb{P} dz \xrightarrow[N \to +\infty]{} 0.$$

Whence

$$\int_\Omega \int \sqrt{\frac{d\mu_\omega}{d\lambda}} dz d\mathbb{P} = 0,$$

and the proof is complete. □

*The reader need not be familiar with Ornstein's construction neither with the spectral theory of dynamical systems.

†Normandy University, University of Rouen Department of Mathematics, LMRS UMR 60 85 CNRS, Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray - France .

[1]It is an easy exercise to check that the Lindeberg condition holds under $dz \otimes d\mathbb{P}$.

★More details.

Notice that

$$\begin{aligned}\int_\Omega \int_{\mathbb{T}} \Big| |P_j(\omega,z)| - |\widetilde{P}_j(\omega,z)| \Big| dz d\mathbb{P} &\leq \int_{\mathbb{T}} \Big| \int P_m(\omega,z) d\mathbb{P} \Big| dz = \int_{\mathbb{T}} \Big| \frac{1}{\sqrt{m}} \sum_{p=0}^{m-1} z^{p(h_m+t_m)} \Big| \Big| \frac{1}{t_m+1} \sum_{s=0}^{t_m} z^s \Big| dz \\ &\leq \Big\| \frac{1}{\sqrt{m}} \sum_{p=0}^{m-1} z^{p(h_m+t_m)} \Big\|_2 \Big\| \frac{1}{t_m+1} \sum_{s=0}^{t_m} z^s \Big\|_2.\end{aligned}$$

The last inequality is due to the Cauchy-Schwarz inequality. This gives

$$\int_\Omega \int_{\mathbb{T}} \Big| |P_j(\omega,z)| - |\widetilde{P}_j(\omega,z)| \Big| dz d\mathbb{P} \leq \frac{1}{\sqrt{t_m+1}} \xrightarrow[m \to +\infty]{} 0.$$

Since

$$\Big\| \frac{1}{\sqrt{m}} \sum_{p=0}^{m-1} z^{p(h_m+t_m)} \Big\|_2 = \Big\| \frac{1}{\sqrt{m}} \sum_{p=0}^{m-1} z^p \Big\|_2 = 1, \text{ and } \Big\| \frac{1}{t_m+1} \sum_{s=0}^{t_m} z^s \Big\|_2 = \frac{1}{\sqrt{t_m+1}}.$$

▶ On the proof of Theorem 5.2. in “Calculus of Generalized Riesz Products” pages 158-162.

The proof of Theorem 5.2. is self-contained and goes as Follows.

- Using Cauchy-Schwarz inequality, we establish that $\sqrt{\dfrac{d\mu}{d\lambda}}$ is a weak limit of the sequence of L^2-functions $\displaystyle\prod_{j=1}^{n} |P_j(z)|$.
- We take advantage of the following formula

  $$\mu = R_n^2(z) d\mu_n,$$

  where

  $$R_n = \prod_{j=1}^{n} |P_j(z)| \qquad \text{and} \qquad d\mu_n = \prod_{j=n+1}^{+\infty} |P_j(z)|^2,$$

  and we prove that any weak limit ϕ of the sequence $\sqrt{\dfrac{d\mu_n}{d\lambda}}$ satisfy $0 \leq \phi \leq 1$. Finally, we deduce that the limit of the sequence $\Big\| R_n - \sqrt{\frac{d\mu}{d\lambda}} \Big\|_1$ is zero.

▶ On the CLT argument.

It is easy to see that Lindeberg’s condition holds for the sequence of the random variables

$$X_m(\omega,z) = \frac{1}{\sqrt{p_m}} \sum_{k=0}^{p_m-1} \Big(z^{n_{j,k}(\omega)} - \int_\Omega z^{n_{j,k}(\omega)} d\mathbb{P} \Big).$$

We thus get that $(X_m(\omega,z))$ converge in distribution to the complex normal distribution under $dz \otimes d\mathbb{P}$.

Acknowledgment 1. *The author wishes to express his thanks to François Parreau, Jean-Paul Thouvenot, Mahendra Nadkarni and Mariusz Lemańczyk for discussions on this note.*
It is also a pleasure for him to express his thanks for the warm hospitality to the organizers of the conference ”Number Theory and Dynamics, 06 July - 10 July 2015, institut Mittag-Leffler”, and to the institut of Mittag-Leffer where this note was revised and completed.

References

[1] E. H. el Abdalaoui and M. Nadkarni, *Calculus of Generalized Riesz Products,* Contemporary Mathematics(AMS) 631 (2014), pp. 145-180. http://fr.arxiv.org/pdf/1307.6513.

[2] J. Bourgain, *On the spectral type of Ornstein class one transformations, Isr. J. Math.* ,**84** (1993), 53-63.